TURKSHEAD & MATHEMATICS (Part 2 of 2)

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Quick summary :

ROTATION (pure) : Points move around the centre of mass with a linear speed proportional to the distance they are from this centre which is itself at rest (its radius = 0).
Points diametrically opposite move in opposite directions.
The direction of linear move makes a right angle with the radius on which the point is placed.
All the points belonging (PP included !) to the circle rotate by a same angle, but their linear speed is a function of their distance from the centre AND from the point of contact. Speed of Tracing Point is varying depending on where it is, being minimum toward the point of contact and maximum when diametrically opposite to this point of contact.

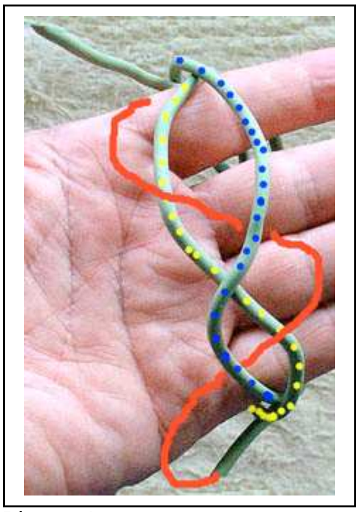
TRANSLATION (pure) : all the points belonging to the RC move in the same direction and with the same linear speed as its centre of mass does.

ROLLING (a mix of the above two motions) : the point of contact is “instantaneously” at rest and behave as an “instantaneous centre of rotation”.
The centre of the RC is then moving at a speed proportional to the radii (on the line : point of contact – moving point) and to the angle that is swept by it.
This goes for any point along the diameter or its continuation (PP holding arm eventually)

Reminder note :
If an observer of the ‘rolling’ moves at the speed that the centre of the circle does, following the same direction, then there is no translation that he can see, there is only a rotation to be observed. A marker fixed on the valve of the bicycle wheel will then be seen as making a circle. If the observer is immobile and the circle is rolling then there is translation ***and*** rotation to be seen :
Drift (translation) + rotation = trochoïd.
The choice of the frame of reference is extremely important. Since Galileo we know motion is “relative”.

A particular ‘family’ , the **3 *LEAD*** THK one :
Why in the 3*L n*B group (n and 3 do not share any prime factor) bights may be added to an existing THK (not one bight only, that is impossible) without having to add any *LEAD* as you must do in the **more than 3 *LEAD*** group for which you have add to *LEAD* to add to *BIGHT* and vice versa ?

Picture 1 shows what exist BUT ALAS DESCRIBING IS NOT EXPLAINING.
When in an existing 3L THK, by “**crossing the bights**”, a supplementary *CUSP* (crossing) is created between 2 pre-existing ones then inescapably 2 empty spaces are created through which the Wend has to pass transforming the so called *BIGHT* it comes from into a *LEAD* going on to expand / enlarge the structure.



Picture 1

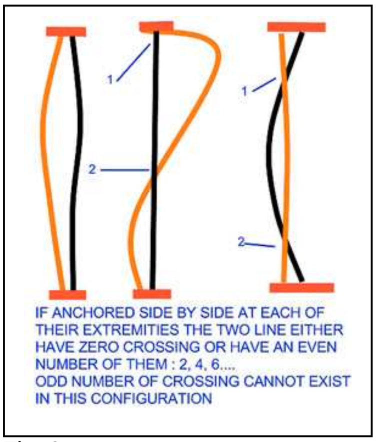


Fig. 9

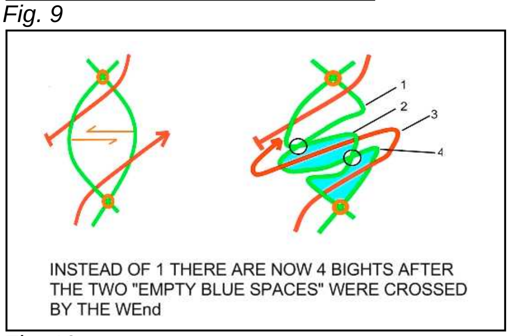


Fig. 10

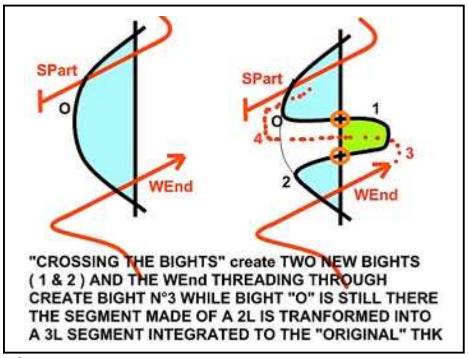


Fig. 11

In fact ONE *LEAD* is then added !
Look at it like that:
making the “***BIGHT* crossing**”, so creating 2 “additional spaces” or “holes”, indeed creates a

small segment of **2** *LEAD* THK included in smooth continuity in the original **3** *LEAD* THK?

Following the *BIGHT* addition threading in the Wend *STRAND* add in fact **1** *LEAD (and 1 BIGHT*) to the **2** *LEAD* segment to get a "coherent homogeneous, continuous" **3** *LEAD* all over.

While there is good reason (speed in the curve, rebroussement) to differentiate ***CUSP*** there is, I think, no real good reason, except Ashley's arbitrary choice to differentiate *BIGHT*.

What has been labelled *BIGHT* is a rather *undifferentiated* part of a *LEAD* (contrary to what the crossing at the cusp is = quite a special point). *BIGHT* sole particularity is to be situated at the frontier, at the border, so it is easily seen ; the tree that hides the forest as they say in France !
The very same segment lying at the frontier is counted twice: once as a *LEAD*, once as a *BIGHT*.
Not what I readily accept as "intellectual strictness".

That is essentially because *BIGHT* don't really exist (only *CUSP*) but are basically of LEAD essence that you can cross the *BIGHT* /*LEAD* in a 3L. You cannot physically do that in a more than 3L. Crossing the *BIGHT* is crossing 2 *LEAD* in fact ; you can do that only if there is no intercalated *LEAD* between them making an obstacle.
Using *CUSP* instead of *BIGHT* immediately allows to perceive what is really happening there.

Using the *CUSP* would have avoided the faulty logic of counting the very same segment under two different headings.

Be as it may this iconoclast thinks there is a snowball's chance in hell, of ever seeing this bit of theology, most sorry…beg your pardon, nomenclature be modified.

It is probably best now to read again part ONE

FAST AND DIRTY RECIPE

RF/GCD or **(RFred)** gives the NUMBER OF ***BIGHT*** (ROTATIONS of RC relative to the FC) ("red" stands for **red**uced)

RR/GCD or **(RRred)** gives the NUMBER OF REVOLUTION (LCM / RF gives that number too)

(RF/GCD) + (RR/ GCD) for **EPI**,
(RF/GCD) – (RR/GCD) for **HYPO** gives the NUMBER OF ***LEAD***, that is the number of rotations of the PP Arm, so of RC, relative to the coordinate system.

Numerical example :
RF = 60 RR = 35 GCD = 5
so RFred = 60 / 5 = 12 RRred =35 / 5 = 7
60 = 2 * 2 * 3 * 5 = 12 * 5 **35 =** 5 * 7
LCM (least common multiple = **2 * 2 * 3** * **5** * **7** = 420 or 60 * **7** = 35 * **12** = 420)
Hence circles ratio (RFred / RRred) = 12 / 7
7 REVOLUTIONS to re-enter the curve
12 *BIGHTS* (12 ROTATIONS of RC relative to FC)

12 + 7 = 19 *LEAD,* respectively 12 - 7 = 5 *LEAD*
During each REVOLUTION, for each *BIGHT* laid, a fraction of *LEAD* is laid, plus (or minus) one *LEAD*.

Number of *LEAD* will be drawn in full only if PP is beyond a certain length .

EXPLANATION OF FAST AND DIRTY

RF / RR (60 / 35) or **RFred / RRred** (12 / 7 e.g) is the number of RC perimeter (a full ROTATION) unrolled on the FC perimeter during one REVOLUTION of RC around FC

For the curve to re-enter, or close on itself, there must be enough REVOLUTION to generate an integer number of unrolled RC perimeters : the length of arc used on RC must be equal to the length of arc used on FC. Number of REVOLUTION = LCM / RF

Let Number of REVOLUTION be NR and number of RC ROTATION relative to FC be nr
To get the integer for re-entry you must have :

RF * NR = RR * nr (60 * **7** = 35 * **12** = LMC)
RF /RR = nr / NR (60 / 35 = **12 / 7**)
here 12 * NR = 7 * nr. It is plain to see that NR must be equal to z*7 and nr to z*12 ; in the numerical example 12 *(z * 7) = 7 * (z * 12)
Hence for z = 1 **NR = RRred and nr = RFred**

- NR and nr are integers sharing no common prime factor, they can be said to be relatively prime.
- nr circumferences of rolling circle will equal NR circumference of the fixed circle. (equality of arcs)
- There are nr CUSP in the complete curve.

Let us see now the RC rotations, the PP Arm rotations too, relative to coordinate system.

The centre of RC travels along a deferent circle of radius **RF + RR** in the case of **EPI** and **RF – RR** in the case of **HYPO** (that is using |values|, if using "signed" values formula is simply RF **+** RR)
To get the number of Arm ROTATION (LEAD laid) per REVOLUTION we divide by RR .

EPI case
(RF + RR) / RR = (RFred + RRred) / RRred = (RFred / RRred) + 1 (angles move in the same direction , same SIGN)
HYPO case
(RF - RR) / RR = (RFred - RRred) / RRred = (RFred / RRred) – 1 (angles move in opposite directions, different SIGN)

RFred + RRred or **RFred – RRred** depending on the case is the total Number of *LEADS*

Numerical example
RFred = 12 [ROTATIONS] RRred = 7 [REVOLUTIONS]
(12 + 7) / 7 respectively (12 – 7) / 7
19 / 7 respectively 5 / 7
The number of *LEADS* laid per revolution is :
19 / 7 = 2.7142 respectively 5 / 7 = 0. 7142

Another view of those 12 BIGHT THK is:
(12 + 7) / 7 = (12 / 7) + 1 **EPI** case
(12 – 7) / 7 = (12 / 7) – 1 **HYPO** case
(12 / 7) **+ 1** ; respectively (12 / 7) **- 1**
(1.7142) **+ 1** ; respectively (1.7142) **- 1**
Each ROTATION of RC relative to *coordinate* lays one more or one less *LEAD* than *BIGHT* were laid in the same REVOLUTION. (12B + 7 = 19L
respectively 12B – 7 = 5L)
Radius of ROTATION is RR, radius of REVOLUTION is RF + RR (respectively RF – RR)

So the impossibility of **L = B** in a single strand THK, strand equivalent to the single continuous line traced out, and the so called common divisor law have always been plain to see, in-built, in those formulas very much older than the grandfather's grandfather of both ASHLEY and TABER.

Another way of understanding why **L = B** is impossible with a single strand THK is :

L = (RF + RR) / GCD or **L = (RFred + RRred)**

B = (RF / GCD) = RFred

L = B imply **RFred + RRred = RFred**
Hence **RRred = ZERO**

RR = ZERO is clearly impossible if a drawing is to exist.
Rolling circle would be an immaterial point on the circumference of the Fixed circle.

2.7142… 0.7142 …
To get the number of *BIGHT* laid per REVOLUTION **RFred / RRred** (e.g 12 / 7 = 1.7142…)
EACH complete ROTATION of RC relative to the FC lays a *BIGHT* (*Fig.3).*
Now a tiny bit of low level formalism.

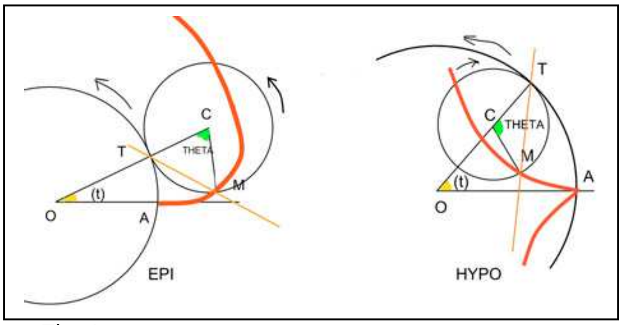


Fig. 1
In this text RF and RR are the ABSOLUTE value of the parameter so RF = |RF| and RR = |RR|. The disambiguation convention of sign minus for the HYPO is directly applied and contained in the formulation. (Using signed values **EPI** would be **+**|RR| and **HYPO** **–**|RR|)
Note about software : there are some that make you chose explicitly "inside", other just let you signify inside (HYPO) by explicitly applying the sign minus to your entry for RC value from the keyboard. (*Fig. 1* above)

EPITROCHOÏD
Initially A, M ,T are at the same place (A = M = T)
ANGLES (t) and Theta are 'linked', linked as in not independent one from the other.

RF / RR = N (N > 1 or RR < RF) RF = N* RR
(you may see RR as being "one unity" worth)
RF * (t) = N * RR *(t) = RR * (Theta)
(measures of arcs and equality of arcs)
Angles (t) and (Theta) are of *identical* sign (angles move in the same direction – PP rotates **ANTI-CLOCK**wise so its absolute value will be signed "plus")
(Theta) = **+** N* (t) = **+** (RF / RR) *(t)

X = RR (N + 1) * cos (t) **–** PP * cos [(N + 1) *(t)]

X = RR ((RF / RR) + 1) cos (t) **–** PP * cos [((RF / RR) + 1) *(t)]

X =(RF + RR) * cos (t) – PP cos [((RF + RR) / RR))*(t)] RF+RR is radius of deferent circle
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Y = RR * ((RF / RR) + 1)* sin (t) – PP * sin [((RF / RR) + 1) *(t)]

Y = (RF + RR) * sin (t) – PP * sin [((RF + RR)/RR) * (t)]

Number of crossing (or of 'holes') = number of BIGHT * (Number of LEAD - 1) = RFred * [(RFred + RRred) - 1] = N of ROTATION comp RF centre * (N of ROTATION comp coordinate – 1) (e.g 12 * (19 – 1) = 216)

HYPOTROCHOÏD
Initially A, M, T are at the same place (A = M = T)
ANGLES (t) and Theta are 'linked' as in not independent one from the other.

RF / RR = N (N > 1 or RR < RF) RF = N* RR
(you may see RR as being "one unity" worth)
RF * (t) = N * RR *(t) = RR * (Theta)
(measures of arcs and equality of arcs)
Angles (t) and (Theta) are of *opposite* signs (angles move in opposite directions – PP rotates **CLOCK**wise so its absolute value will be signed "minus")
(Theta) = **-** N* (t) = **-** (RF / RR) *(t)

X = RR (N - 1) * cos (t) **+** PP * cos [(N - 1) *(t)]

X = RR ((RF / RR) - 1) cos (t) **+** PP * cos [((RF / RR) - 1) *(t)]

X =(RF - RR) * cos (t) + PP cos [((RF - RR) / RR))*(t)] RF-RR is radius of deferent circle
----
Y = RR * (N – 1) * sin (t) – PP * sin [(N – 1) *(t)]
Y = RR * ((RF / RR) - 1)* sin (t) – PP * sin [((RF / RR) - 1) * (t)]
Y = (RF - RR)* sin (t) – PP * sin [((RF - RR) /RR) * (t)]

Number of crossing (or of 'holes') = number of BIGHT * (Number of LEAD - 1) = RFred * [(RFred - RRred) - 1] = N of ROTATION comp RF centre * (N of ROTATION comp coordinate – 1) (e.g 12 * (5 – 1) = 48)

HOW TO SIGN THE VALUES

(“0” of X and Y axis is the centre of FC)

EPI

RF = **+** |RF|
RR= **+** |RR| as centre of RC is OUTside FC perimeter
PP = **+** |PP| if Pen Point is on the right of RC centre
PP = 0 is Pen Point is at the centre of RC
PP= **-** |PP| if Pen Point is BETWEEN the 2 centres or on the left of centre of FC and nearer FC centre than RC’s

HYPO

RF = **+** |RF|
RR= **-** |RR| as centre of RC is INside FC perimeter
PP = **+** |PP| | if Pen Point is on the right of RC centre
PP = 0 is Pen Point is at the centre of RC
PP= **-** |PP| if Pen Point is BETWEEN the 2 centres or on the left of centre of FC and nearer FC centre than RC’s

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ANOTHER VIEW OF THE FORMULAS (values used here are ‘absolute’)

Equations can also be written using
PP = K * RR and replacing PP by K * RR
If K = 1 === EPI and HYPO**CYCLOÏDS**

The equations : (every thing is aligned along axis X at the start position) can be seen as :
Rolling = Translation + Rotation

Epitrochoid case :
x = (RF + RR) cos(t) - PP cos(((RF + RR)/RR)*(t))
y = (RF + RR) sin(t) - PP sin(((RF + RR)/RR)*(t))

Hypotrochoid case :
x = (RF - RR) cos(t) + PP cos(((RF - RR) / RR)*(t))
y = (RF - RR) sin(t) - PP sin(((RF - RR) / RR)* (t))

Remember the equality of ARCS ?
(RC * t) + (RC * Theta) = RF * t
RC * Theta = t * (RF - RC)
Theta = t * [(RF - RC) / RC]

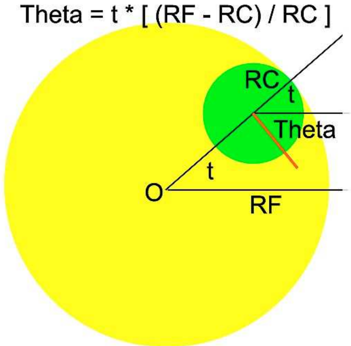

Fig. 2

The above equations can be written as :

- **Epi**trochoid :
x = (RF + RR) cos(t) - PP cos(Theta)
y = (RF + RR) sin(t) - PP sin(Theta)

-**Hypo**trochoid :
x = (RF - RR) cos(t) + PP cos(Theta)
y = (RF - RR) sin(t) - PP sin(Theta)

RF **-** RR or RF **+** RR express the radius of the DEFERENT circle along the perimeter of which the centre of RC translates respectively in the **HYPO**trochoid and in the **EPI**trochoid.

PERIODICITY

If **RF / RR** is **RATIO**nal - it can be written as a RATIO of integers not sharing any prime factor – except 1 - then the period is the numerator RF expressed as its own IRREDUCIBLE FRACTION (RF_{red} = number of BIGHT)
RFred and **RRred** share no common prime factor.

Periodicity means the curve will re-enter, meet itself ‘exactly‘. A ‘round’ number of rotations of RC will bring the tracing point to its exact starting point, so closing the tracing.

RC regains its exact original starting position and orientation after having made RR_{red} *revolutions* around the FC and the RC will have accomplished RF_{red} / RR_{red} *rotations* on itself relative to RF centre per *revolution*. (*BIGHT* laying)

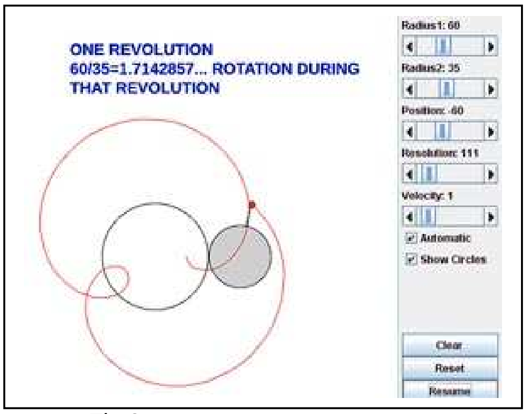

Fig.3

(RF + RR) / RR, (RF - RR) / RR is the number of ROTATION relative to coordinate made by RC around FC in one REVOLUTION (*LEAD* laying)

(RF + RR) / RR = (RF / RR) + 1 =
(RFred + RRred) / RRred = (RFred / RRred) + 1

(RF - RR) / RR = (RF / RR) – 1 =
(RFred - RRred) / RRred = (RFred / RRred) – 1

LEAD laying = *BIGHT* laying **+** 1 (respectively **-**1)
This is the number of *LEAD* laid during <u>each</u> REVOLUTION.

RF / GCD, that is **RFred,** is the number of **CUSP,** or of *BIGHT* . [e.g RF = 60 = **2*2*3***5 RR = 35 = 5***7**]

RR / GCD, that is RRred, is the number of REVOLUTION the curve need to "re-enter".
RFred + RRred is the Number of *LEAD* for EPItrochoide (absolute |values| being used)
RFred - RRred is the Number of *LEAD* for HYPOtrochoid (|values| being used)

Total *LEAD* laying = Total *BIGHT* laying + Number of REVOLUTION (respectively -)

Please do experiments with different PP values.
example as before :
EPI RF = 60 RR = 35 or +35
HYPO RF = 60 RR= -35 .

Try : (use both absolute and signed values)
PP=RR
PP > RR but PP < RF + RR
PP = RF + RR
PP > RF + RR
PP >= (2 * RF) + RC

Do it too for HYPOtrochoid.
Note : you have to use the MINUS SIGN for entering RR from the keyboard in some software , in some others use the 'inside' setting with 'absolute' values.

In my experience the most complete curves are obtained with PP = 2 * RF + RR (EPI case)
Try also PP = (3/2*RF) + RR EPI case or with (-RR) for HYPO case

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Reminder :

	0 (radian) or degree)		Pi/2 or 90°		Pi or 180°		3*Pi /2 or 270°		2*Pi or 360°
COSinus	+1	waNing	0	waNing	-1	waXing	0	waXing	+1
SINus	0	waXing	+1	waNing	0	waNing	-1	waXing	0

In trigonometry, angles are signed PLUS in anti-clockwise rotation and MINUS in clockwise rotation.
This is just a disambiguating *convention*.
It is not a measure of concrete reality.

Note that :

- for **EPI**trochoids ROTATION and REVOLUTION are in the **SAME** direction : either both anti-clockwise or both clockwise

- for **HYPO**trochoids ROTATION and REVOLUTION are in **OPPOSITE** directions : one is clockwise while the other is anti-clockwise.

The formulas suppose that at the start all is aligned along the positive X axis.

(t) is the angle of rotation in **radians**.

Term (t) is *linear* motion

Terms Sin (t) and Cos (t) are *angular* motion

EPItrochoid = The RC rotating in one direction, the relative rotation of the PP arm rigidly fixed on it has the same direction.

The total number of rotation of PP arm *relative to the coordinate system* will be **(RF / RR) + 1**
Think sidereal versus solar. (per REVOLUTION)
Number of ROTATION relative to *coordinate* =
Number of ROTATION relative to FC *centre* **+ 1**.

HYPOtrochoid = since the relative rotation of the PP arm is opposite the direction the circle is rolling, the *absolute* number of the arm rotation as measured by the *coordinate system* is one less than the number of rotations made by the mobile around the centre of the fixed. **(RF / RR) - 1**

EPItrochoid
X = (RF + RR) Cos (t) - PP Cos [((RF / RR) + 1)t]
Y = (RF + RR) Sin (t) - PP Sin [((RF / RR) + 1) t]

HYPOtrochoid
X = (RF - RR) Cos (t) + PP Cos [((RF / RR) - 1) t]
Y = (RF - RR) Sin (t) - PP Sin [((RF / RR) - 1) t]

DOUBLE GENERATION (Luigi CREMONA)
Here are 2 links as illustration only.
http://poncelet.math.nthu.edu.tw/chuan/99s/4/rose-d.htm

http://steiner.math.nthu.edu.tw/disk3/gc-03/8/3d.html

LOOKING ANEW INSIDE EQUATIONS

The centre of RC travels around FC following the perimeter of a DEFERENT circle with radius :
RF + RR for **EPI**trochoid
RF – RR for **HYPO**trochoid

The parametric equations for this centre of RC are :
(case of **HYPO** only here)
x = (RF – RR) * cos (t)   y = (RF – RR) * sin (t)

This is *not* the **HYPO**trochoid itself : the parametric equations of the Tracing Point (which being rigidly fixed on the RC is subjected to the motion of translation of RC centre) must be included.
They are
x = PP * cos (Theta)   y = PP * sin (Theta)
that gives
x = (RF – RR) * cos (t) + PP * cos (Theta)
y = (RF – RR) * sin (t) + PP * sin (Theta)

Now we 'homogenise' by replacing (Theta) by its value expressed in term of (t)
Let us make it a short story
Theta= ((RF-RR) / RR) * (t) = (RF/RR -1) * (t)

End result is

for **HYPO** case :
x = (RF – RR) * cos (t) + PP * cos [(RF/RR -1) * (t)]

y = (RF – RR) * sin (t) - PP * sin [(RF/RR -1) * (t)]

for **EPI** case :
x = (RF + RR) * cos (t) - PP * cos [(RF/RR +1) * (t)]

y = (RF + RR) * sin (t) - PP * sin [(RF/RR +1) * (t)]

HARD AND FAST EQUATIONS

for **both EPI** AND **HYPO** cases, under the provision that "**SIGNED**" values are used in them (only SIGN "**+**" is used in the equations).[Reminder :
a * (-b) = (-a) * b = -ab and a /(-b) = (-a)/b = **-** (a/b)
a **+** (-b) = a **-** b and a **-** (-b) = a **+** b]

$X_{(t)}$=(RF + RR) * cos (t) + PP * cos [((RF + RR) / RR) * (t)]

$Y_{(t)}$=(RF + RR) * sin (t) + PP * sin [((RF +RR) / RR) * (t)]

PRACTICAL EXAMPLE

60 / 35 = 1 remainder 25
(60 * 2) / 35 = 3 remainder 15
(60 * 3) / 35 = 5 remainder 5 …/… 60 * 4
(60 * 5) / 35 = 8 remainder 20 …/… 60 * 6
(60 * 7) / 35 = **12** remainder **ZERO** so 12 REVOLUTION

shorter it is to make use of the LMC as in
LCM / RR = (2 * 2 * 3 * 5 * 7) / (5 * 7) = 2 * 2 * 3
420 / 35 = 12 ROTATION

12 RC rotations imply 12 * RR (12 * 35 = 420) as representing arc length - in fact it is
12 * (35 * 2 * Pi) = 420 (2 * Pi) -
so 420 / RF 420 / 60 = 7 REVOLUTION - in fact it is
(12 * 35 * 2 * Pi) / (60 * 2 * Pi) = (12 * 35) / 60 = 420/60

A shorter way is LCM / RF = 420 / 60 = 7
(2 * 2 * 3 * 5 * 7) / (2 * 2 * 3 * 5) = 7

If you insist on GCD
GDC = 5 [RF = 60 = 2 * 2 * 3 * **5** RR = 35 = **5** * 7]
RF / GCD = RFred 60 / 5 = 12 ROTATION
RR / GCD = RRred 35 / 5 = 7 REVOLUTION
12 * RC perimeter = 7 * RF perimeter (using equality of arcs).

Or use directly the CIRCLES RATIO (RF / RR) (60 / 35) and express it as an irreducible fraction (RFred / RRred) or (12 / 7)
(RFred) is Number of ROTATION (12)
(RRred) is Number of REVOLUTION (7)
