

## TURK'S HEAD & MATHEMATICS (part 1 of 2)

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- Why a single strand THK cannot have a number of BIGHT equal to the number of LEADS ?

(By THK I mean the ONLY TRUE THK -as far as I am concerned- as exposed in a previous KN issue)

or : why in a single strand THK numbers of BIGHT & LEAD must not share a common prime factor (1 excepted), why do they have to be "relatively prime"?

- Why TURN, old appellation, seems ( to me at least ) more knowledgeable about the deep structure of the knot than BIGHT ( PLAT , SCALLOP, CROSS ) ?

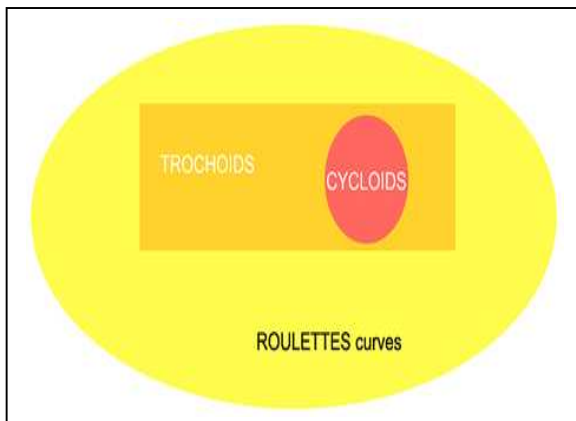
Though no mathematician by profession, I decided to try to simplify (egregious idea it is to dyed in the wool mathematicians) the maths and, in the manner of 19<sup>th</sup> mathematical vulgarisation, to use mainly words, illustrations to see what sort of mathematical creatures THK are.

For expediency we will limit ourselves to the case of THK in mat form, lying flat on a plane.

I tend to find 2D equations less troublesome to explain without going deep in mathematics than 3D cylindrical or elliptic (spherical ) are.

Yet as you already know for having gone from one form to the other in knotting flat THK are strict topological equivalent of the cylindrical varieties.

**THK are TROCHOÏDS.**



CYCLOÏDS are special cases of TROCHOÏDS which themselves are special cases of « **ROULETTE** ».

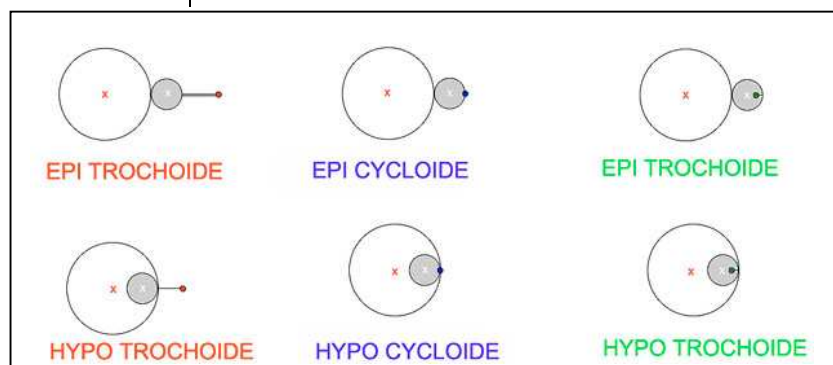
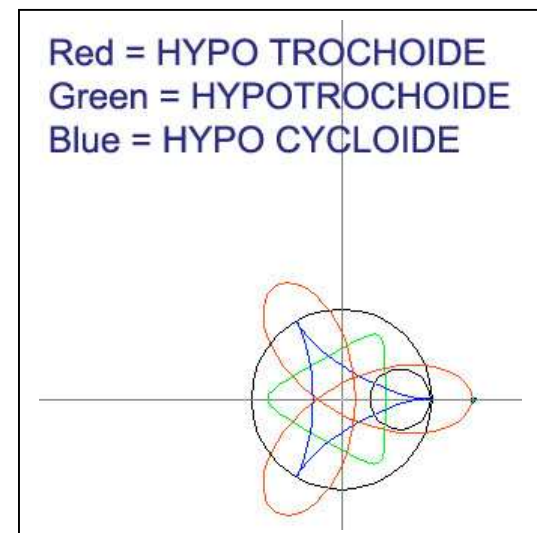
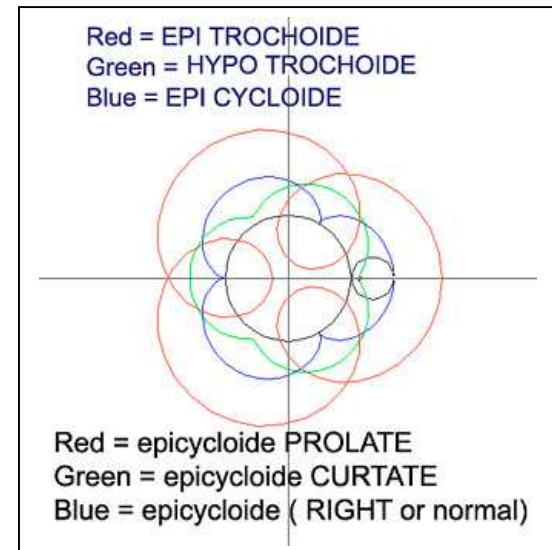
You may read here and there that TROCHOÏDS are special cases of CYCLOÏDS. Some authors have totally abandoned TROCHOÏDS and speak only of CYCLOÏDS.

My view is : Cycloïds are but trochoïds with this particularity that the tracing point is exactly on the perimeter of the rolling circle. (my sources are 17<sup>th</sup> to 20<sup>th</sup> centuries European / French publications)

Trochoïds come in two flavours : **EPI**, that is ABOVE and **HYPO** that is BELOW.

EPITROCHOÏDS and HYPOTROCHOÏDS.

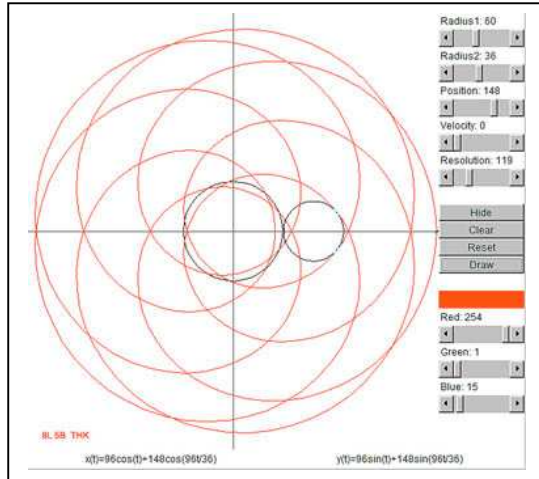
Here are the curves.



**Material : Two circles, one FIXED, one MOBILE.**  
The MOBILE circle is also known as the **GENERATING** circle. Its motion trace out the curve.

At this stage may be some of you have yet to see how this can be related to a THK template.  
Just have a peek :

Fig. 1



Is this not a template for a 8L 5 B THK ?  
This software draws trochoid and cycloid curves.

It is time to built the necessary common ground making some clarification :

- **FIXED** circle (**FC**) : Astonishingly this circle does not move ! It is centred at the 'origin' of X and Y. Its perimeter is the guiding line for the rolling circle (**RC**) which accomplish revolution(s) around it. Note that if the radius of this FC is infinitely enlarged its perimeter becomes a straight line, as in the case of classical cycloid curves.

- **MOVING** or **ROLLING** circle is the mobile circle to which the arm holding the tracing point that **GENERATES** the tracing is rigidly attached.

**Pen Point** Holding Arm will be abbreviated into **PP**.

The parameters are the perimeters of the circles.  
A circle perimeter is in direct proportion to the radius as in  $(2 * \pi * R)$ , so radii are used instead.

**RF** will be the **R**adius of the **F**ixed circle (**FC**)  
**RR** will be the **R**adius of the **R**olling circle (**RC**).

- **ROTATION** : this is a circular (angular) motion, around an **internal axis** belonging to the figure or mass doing the spinning. ( spinning top )  
The Earth undergoes a rotation on itself, spinning on its own axis. The span of time in which Earth accomplish a full rotation on itself is "a day". (we will see that there are two ways to count a full Earth **ROTATION** : one *sidereal*, the other *solar*)

- **REVOLUTION** : this is turning around, or *travelling* around an **external axis**. A sort of orbiting course.

The Earth accomplish a full **REVOLUTION** around the Sun in the time span of a year, all the while rotating on itself.

- **ROLLING without slippage** amount to adding a **TRANSLATION** motion (moving around) to a **ROTATION** motion (gyration). When car wheels are only rotating and not making any translation then the car is mired. On a dry ground they rotate around their axis and make a translation : car moves along.

A **translation** is moving the centre of mass of the figure, here the centre of the RC.  
There is one **REVOLUTION** when the path close on itself. This happen when the RC perimeter has rolled the exact length of the FC perimeter.

A **rotation** : RC spins around an internal axis, passing through its centre, perpendicular to the circle plane. Again, two types of reckoning are to be taken in account when dealing with the rotation of RC.

The Pen Point arm **ROTATES** while the centre of the RC to which it is rigidly attached **TRANSLATES**. (same or opposite direction as is the case, this is a point to bear in mind)

The *distance from the centre of RC to the tracing point* is abbreviated as PP in formulas.

- **Greatest Common Divisor** (GCD) aka **Greatest Common Factor** or Highest Common Factor.  
Taking 2 numbers (a, b), each different from zero, their GCD is the largest positive integer that can divides (a) and (b) without leaving a remainder (modulo result is zero that is or even division – even if the result is odd ;-))  
We could have use the route of **LCM** (**Least Common Multiple**) instead, but due to the so-called common divisor law it will be the GDC.

## THE CURVE

The epitrochoid ( epicycloid, hypotrochoid ) is the curve traced by a point rigidly fixed on a circle that rolls without sliding on the perimeter of a fixed circle ( situated in the same plane ) ; the perimeter of the rolling circle being in external or internal contact with the perimeter of the fixed circle.

The characteristics of the curves that the RC generates are under the influence of the radii of the two circles. (and PP length of course).

The tracing point itself can be inside, on the perimeter or outside of the RC.

Depending on PP different shapes of curves are traced out with a given couple of radii.

Note that if  $RR > RF$  and RC is touching the outside of the fixed circle 2 configurations can arise : the RC and the FC are outside each other **or** the **FC** is **inside** the **RC**. (external or internal RC contact)  
This latter case is in French called péritrochoïde.  
<http://www.mathcurve.com/courbes2d/peritrochoid/peritrochoid.gif> Equivalent to **EPI**trochoid  
We will leave alone the **DOUBLE GENERATION**.

(Java enabled browsers - in my experience Firefox or Opera are better than Internet Explorer)

<http://www.wordsmith.org/~anu/java/spirograph.html> very fast and makes for swift trail blazing but not good enough IMO for experimenting in depth

<http://www.math.psu.edu/dlitttle/java/parametricequations/spirograph/SpiroGraph1.0/index.html> Very good

<http://www.math.duke.edu/education/ccp/materials/mvcalc/spirograph/spirograph.html> The clean version of the above

<http://www.math.psu.edu/dlitttle/java/parametricequations/spirograph/index.html> The modern version of the above

<http://www.coolmath.com/coolthings/amazing Spiro/spiro/index.html> very good and you can choose your screen size but you don't see the circles in action.

<http://libadis.com/java/spirograph.html>

<http://libadis.com/java/realspirograph.html> this one reproduce the « real » spirograph™ sold in box with all its restrictions.

<http://www.recursoes.pnte.cfnavaarra.es/~msadaall/geogebra/trocoides.htm> this one is just good for a first intro. Very limited so very simple to use.

<http://scratch.mit.edu/projects/rof/18082#> You get the choice of RF ( R1) and RR (R2) and it is directly expressed in what will be Bight and Lead. Not useful beyond being a first intro.

### A LITTLE LATERAL EXPLORATION

should help us with ROTATIONS and REVOLUTIONS in trochoid tracing, *BIGHT* and *LEAD* in THK.

There is a discrepancy of ONE DAY between the **SIDEREAL** (reference is Sun-centred, stars in practice) year and the **SOLAR** (reference is on the Earth surface) year (1 year = one complete REVOLUTION of the Earth around the Sun - about 4mn of difference between the sidereal day and the solar day).

Earth solar day, or daily Earth rotation on itself is measured *compared to the Sun*.

Sidereal is measured *against a coordinate system* made by far away stars.

One can give the « go » and « stop » for counting a full rotation either relative to the Sun or relative to the Stars.

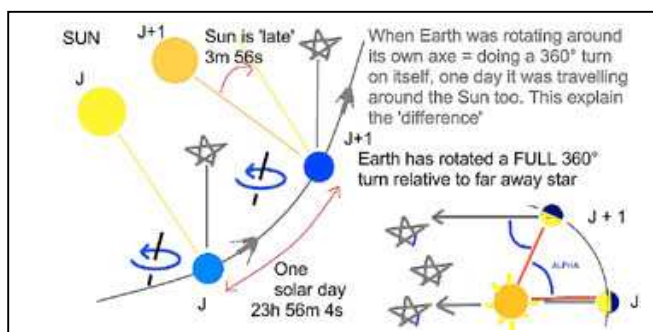


Fig. 2

We could have chosen the Moon sidereal and synodic periods to get at the same phenomena, only it is more complicated.

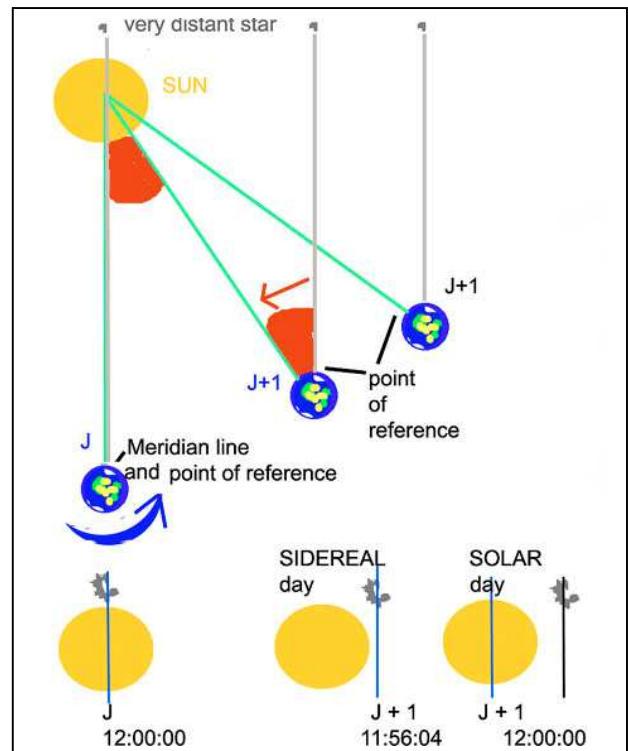


Fig. 3

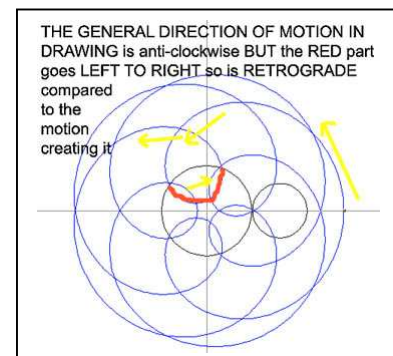


Fig. 4

These curves were used by astronomers to explain the apparent retrograde motion of planets some millenaries before Galileo.

In French, trochoïds curves are qualified by their number or "rebroussement", equivalent of 'cusps'. Rebroussement is "going back".

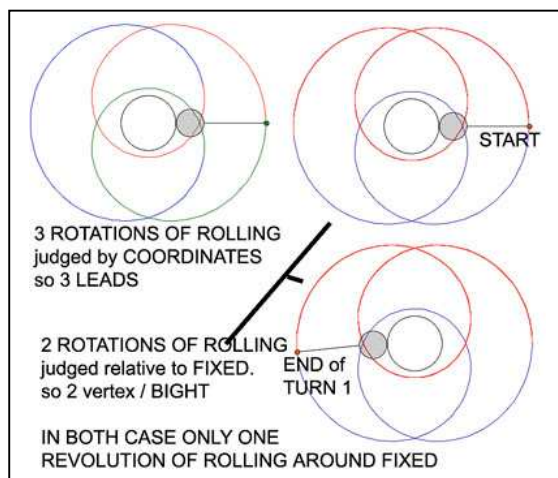
A picture of a planetary retrograde motion is available at :

<http://www.megalithicsites.co.uk/images/retrograde.gif>

### BACK TO TRACK or rather to TROCHOÏDS

With images only, not using any mathematics formulas, let us study the ROTATION of the RC spinning about its own axis

Fig.5



The RC spun three times relative to coordinate (3 days : red, blue, green) in one « year » or one full REVOLUTION of its centre around the FC.

Result = 3 LEAD

Relative to the coordinate system means that the PP Arm gets back in an identical alignment compared to X and Y axis as it had at the start.

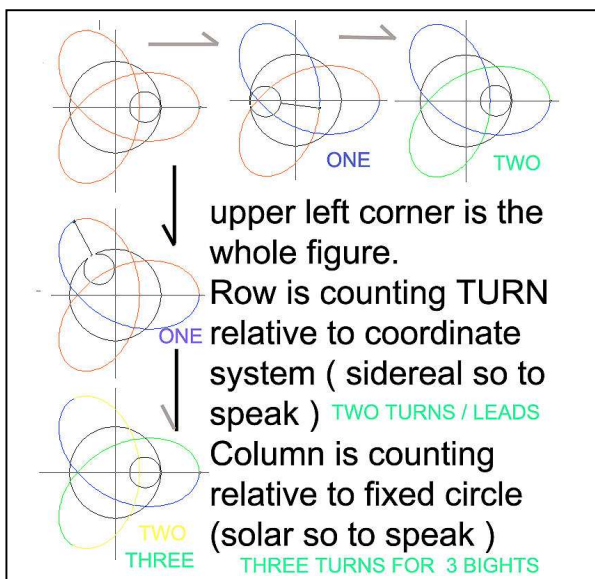


Fig. 6

Rotations (days) drawn on the right side of Fig.5 are counted relative to the FC.

Each time that, relative to the centre of the FC, the RC gets into an orientation identical to its start position orientation the one « turn », one rotation of the RC has been accomplished in full.

Result = 2 BIGHT.

One rotation relative to the FC means that the RC presents itself under the same perspective to an observer placed at the FC centre.

It is important to get the correct feeling about the 2 types of rotations, on how they differ from each other and how they both differ from a revolution.

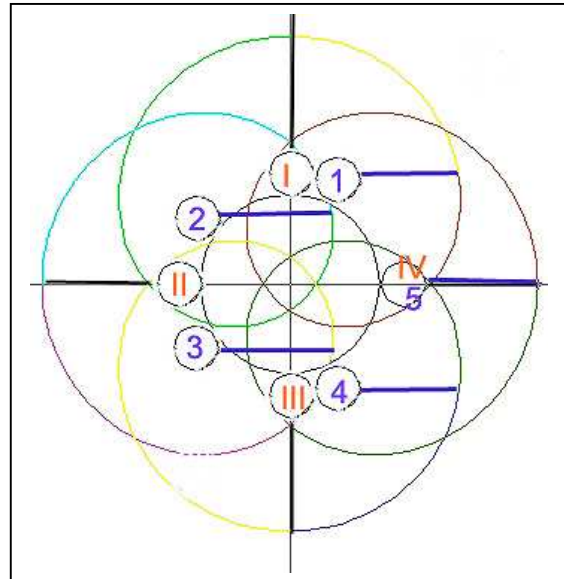


Fig. 7

Because 'rolling' is a composite motion it must be viewed in two frames of reference : coordinate system and FC.

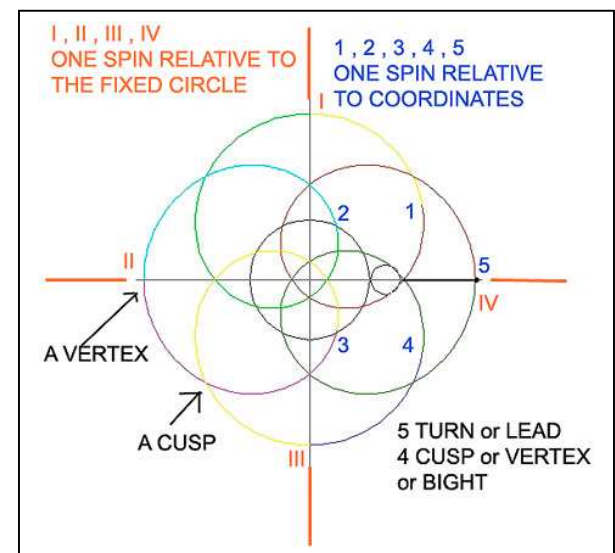


Fig. 8



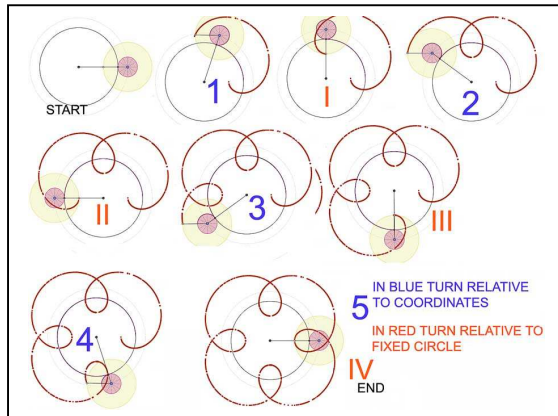


Fig. 9

After studying those images it should now be plain how **BIGHT** and **LEAD** are formed under the dependence of different types of ROTATION having different radii ( RF+RR and RR ) & angular speed.

Also easy to see :

\* the difference between ROTATION ( the spinning of the RC about its own axis ) and

\*\* REVOLUTION ( ultimately linked to TRANSLATION of the RC centre around the FC centre ) .

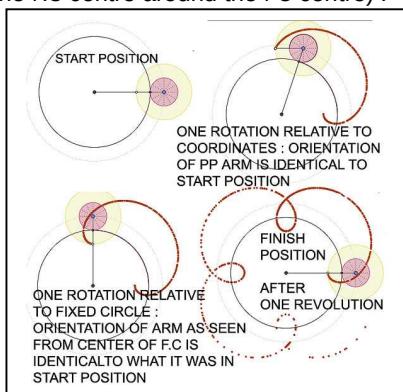


Fig. 10

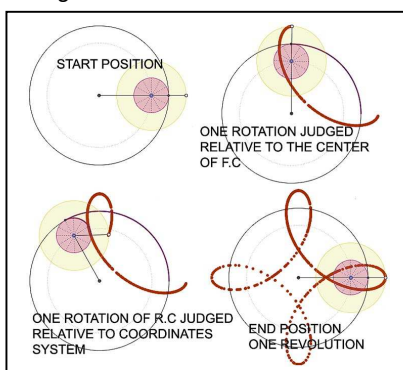


Fig. 11

Note that in the case of **EPI**trochoid « relative to coordinate » is quicker than « relative to FC » ; for the **HYPO**trochoid it is the opposite. ( remember: same direction / opposite directions of rotation )

In **EPI** the arc length used on the RC for 'relative to coordinate' is less than the arc length used on the RC for 'relative to FC'. It is the opposite for **HYPO**.

\*\*\* the difference between « turns » made by the Pen Point Arm counted compared to the coordinate system using a fraction of the RC perimeter, and « turns » made by the circle measured relative to the centre of the FC and using one full perimeter of the RC is also to be noted.

Of course using equations would have been more rigorous...but I fear that we would all have found them rather a chore at this point. Wait for Part 2.

Number of **BIGHT** is governed by the « turns » counted compared to the fixed circle while the Number of **LEAD** is governed by the « turns » reckoned compared to the coordinate system.

Now do you get an inkling about why there can be no THK with Number of **BIGHT** = Number of **LEAD** ?

Do you see why I believe that « **TURN** » ( used in a 1934 book « *Knots, Splices & Fancy Work* » by Chas . L. **SPENCER** page 122), rather than «**BIGHT**» is more adapted to the true nature of the THK. **SCALLOP** was better too than **BIGHT**.

P.P.O **HARRISON** in 1964 used « **TURN** » for **LEAD**.

One last summary in image (**EPI** case ):

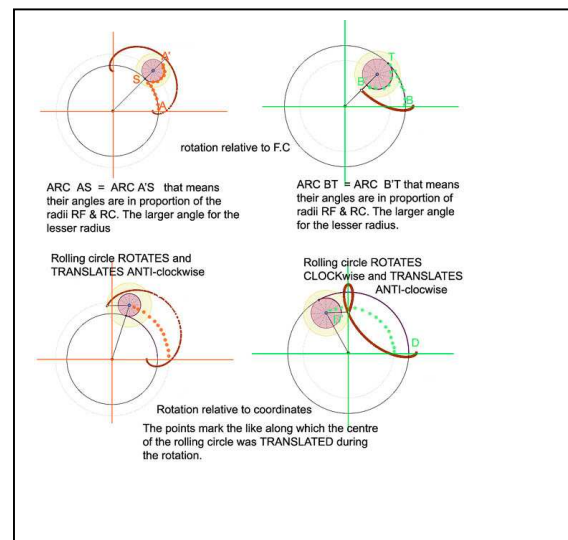


Fig. 12

When the curve re-enter :

- **EPI** case (with  $RR < RF$ ) :

there are more RC rotations relative to coordinate than rotations relative to FC. More **L** than **B**

-**HYPO** case (with  $RR < RF$ ) :

there are less RC rotations relative to coordinate than rotations relative to FC. Less **L** than **B**

**Caveat** : Reader beware ! Despite the careful screening I made, at each step please do verify that I have not made a 'typo' or worse, had a 'neuronal short-circuit'.

A small bit of mathematical nomenclature (illustrations mainly) that should help us explore the mathematics of trochooids at daisies's level.

Fig. 13

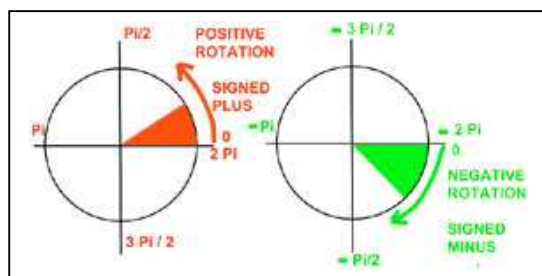
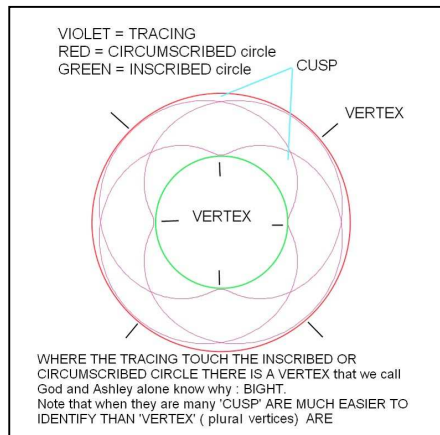


Fig. 14

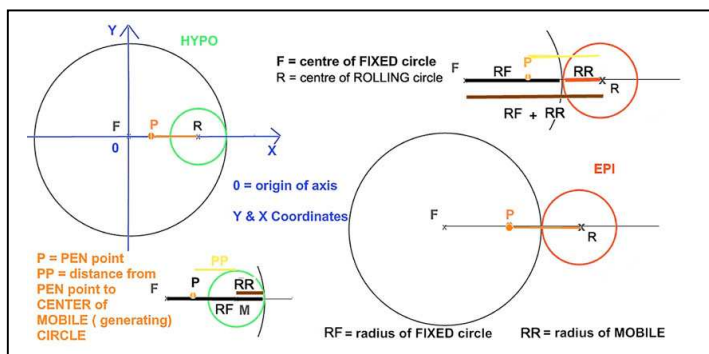
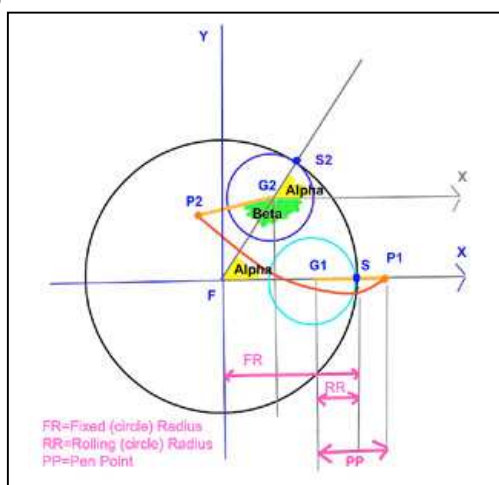


Fig. 15

Fig. 16



**Angle Alpha** (Fig.16) is the angle related to REVOLUTION or TRANSLATION

**Angle Beta** (Fig.16) is the angle related to ROTATION (of RC hence the rotation of the Pen Point which is rigidly fixed on it)

If  $RF < RR$  then Angle Alpha can never be equal to Angle Beta existing at the same time, **so there is no possibility of Number of BIGHT equal to Number of LEAD (TURN)**. Illustrated Fig.16 is the **HYPOtrochoid**, but this is valid for **EPItrochoids**.

If the tracing point is at the centre of RC then a circle is drawn.

That is, if you want to see it as such, a case of equality **ONE BIGHT, ONE LEAD**, which is the "NULL THK" so to speak. Another "NULL THK" with RR inside RF and  $RR = RF/2$  is the ellipse. Vertical or horizontal ellipse depending on the sign of PP. If RF gets very small compared to a very large RR and  $PP < 0$  then a spiral happen that change direction. (please do the experiment)

**Please note** ( Fig. 17 ) that the perimeter of the **green** circle on which the RC centre translates has a shorter length ( **radius = RF - RR** ) than the **red** circle perimeter has ( **radius = RF + RR** ). So, for a given angular motion, a RC of a given radius will translate faster on the green than on the red.

Both green and red circles, have  $2\pi$  radian as 'total angle' **\*but\*** remember that angle is the ratio between the radius and the length of arc sustained by the angle. Hence for an equal angle, the one with the shorter radius will have the shorter arc.

This is important, in reference to ROTATION relative to coordinate system, hence in relation to **LEAD** laid per REVOLUTION.

Make absolutely sure you remember these **GREEN** and **RED** deferent circles at the time mathematics will be spoken of !

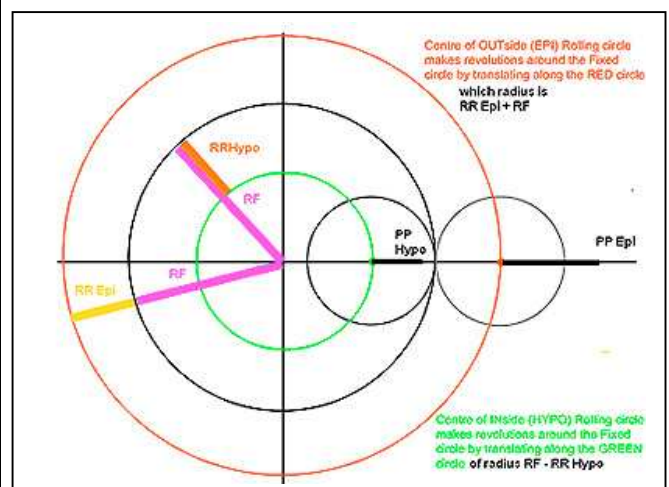


Fig.17

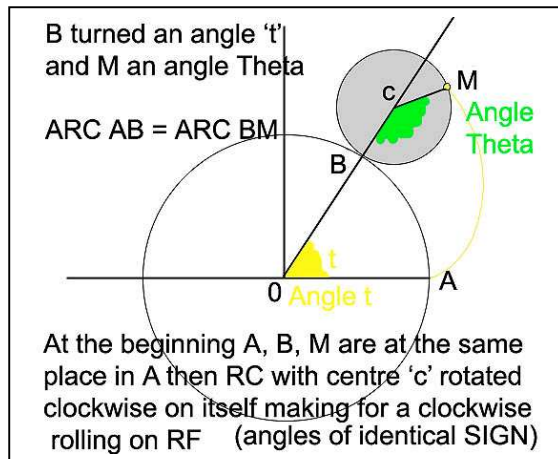


Fig. 18

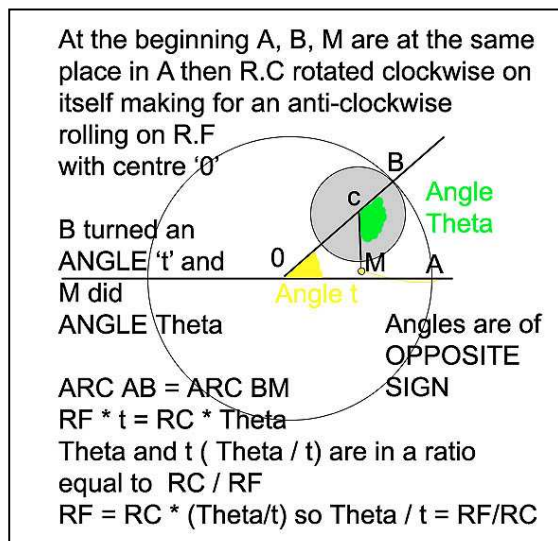


Fig. 19

The equality of the arcs is an important point. The ratio of radii makes the ratio of angular motions. The greater the  $RF / |RR|$  ratio ( circles ratio ) the greater the  $|\Theta \text{ angle}|$  will be.

Another important point is the direction of the angular motions relative to coordinate : in the same direction or in opposite ones (this governs the sign + or - that will be used in the formulas).

If the circle is rotating anti-clockwise then the angular speed is "by convention" considered positive. (clockwise is signed as negative). (note : speed is physics ; roulettes are geometry) **Angular velocity (rotation) is measured relative to the coordinate system while linear velocity (translation) is measured relative to FC centre.**

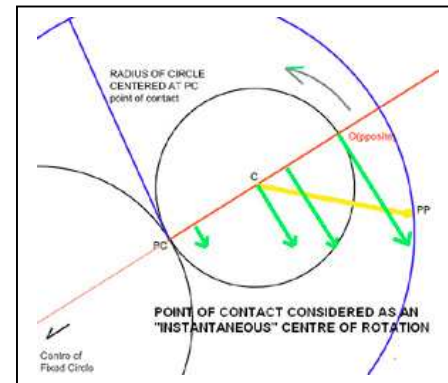
Note that a 'negative' speed does not change the actual movement nor its physical value.

It is simply a convention, a bit like a standardised answer to a logical test : " are the directions concordant ? " ; affirmative answer or 'TRUE' is given sign PLUS and negative answer , or 'FALSE', (opposite directions) is given the MINUS sign. This is just disambiguation device so to speak.

The velocity or speed setting in softwares act on the speed of execution of drawing. Slow is easier to see.

The velocity of PP is a composition of the angular motion  $t$  and of the angular motion  $\Theta$ .

In a rotational motion the speed of a given point along a radius is proportional to the distance



between that point and the centre : farther is faster.

Fig. 20

What does 'without slippage' mean ?

Let us consider the point of contact between the 2 circles. When examined in a very short span of time, 'instantaneously', this point of contact is seen 'at rest', with a speed of ZERO. ( Fig. 20 )

This 'instantaneously immobile' point may be seen as the "instantaneous centre of rotation" for all the points situated on the RC diameter passing through this point of contact.

The point diametrically opposite to this point of contact is the point with maximal speed.

So the Tracing Point will have its maximal velocity each time the tracing diameter ( diameter passing through Tracing point ) is in line with one of RF diameter and the PP is the farthest possible distance from the point of contact.

Without slippage means that the point of contact is always changing in very minute spans of time.

A translation movement is a translation of the centre of mass ( here centre of circle ) plus the translation of all the points of the circle that move in the same direction, at the same speed this centre does. (the bicycle moves/ translates as a whole, "a block".)

Quite a difference from what happen in a rotation. All points attached to the RC will then have the same angular speed \*but\* will follow different paths, at different linear speed. Fig. 20

The hub of a bicycle wheel and a point taken on the rim will cover an equal linear distance as measured on the road, but as seen from an observer immobile on the ground, the hub of the wheel rolling on the flat ground will go in a straight line parallel to the road , while the point on the rim travels a circle arc. In the case of trochoid the equivalent of the straight line parallel to the road is the 'deferent' circle (in red and in green in Fig. 17

END of PART ONE